

# HIGH DISTANCE HEEGAARD SPLITTINGS VIA FAT TRAIN TRACKS

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**ABSTRACT.** We define *fat* train tracks and use them to give a combinatorial criterion for the Hempel distance of Heegaard splittings for closed orientable 3-manifolds. We apply this criterion to 3-manifolds obtained from surgery on knots in  $S^3$ .

## 1. INTRODUCTION

In this paper we derive some combinatorial tools which are useful in determining estimates for the distance of Heegaard splittings of closed oriented 3-dimensional manifolds. In particular, we prove:

**Theorem 5.3’.** *For any  $n \geq 0$  there exist a knot  $k \in S^3$  and an integer  $m_0$ , such that for any  $m \in \mathbb{Z} \setminus \{m_0, m_0 + 1, m_0 + 2, m_0 + 3\}$  the 3-manifold  $M(m)$ , obtained by  $\frac{1}{m}$ -surgery along a horizontal slope on  $k$ , admits a Heegaard splitting  $M(m) = V \cup_{\Sigma} W$  which satisfies*

$$d(V, W) \geq n.$$

Here  $V \subset M(m)$  and  $W \subset M(m)$  are handlebodies which intersect in  $\partial V = \Sigma = \partial W$ . Each of them,  $V$  and  $W$ , determines an infinite subcomplex  $\mathcal{D}(V)$  and  $\mathcal{D}(W)$  in the curve complex  $\mathcal{C}(\Sigma)$ , and the distance  $d(V, W)$  is defined as minimal distance in  $\mathcal{C}(\Sigma)$  between a vertex  $D \in \mathcal{D}(V)$  and a vertex  $E \in \mathcal{D}(W)$ . The curve complex  $\mathcal{C}(\Sigma)$  and its metric  $d$  are defined at the beginning of Section 2.

The main combinatorial tool, to prove this result, is the following theorem, which is derived from some basic observations that can already be found in the work of Masur-Minsky [11], such as in particular the use of nested train track towers.

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**Theorem 4.7.** *Let  $M$  be an oriented 3-manifold with a Heegaard splitting  $M = V \cup_{\Sigma} W$ . Consider complete decomposing systems  $\mathcal{D} \in \mathcal{CDS}(V)$  and  $\mathcal{E} \in \mathcal{CDS}(W)$  which do not have waves with respect to each other. Let  $\tau \subset \Sigma$  be a complete fat train track with exceptional fibers  $\mathcal{E}_{\tau} = \mathcal{E}$ , and assume that  $\mathcal{D}$  is carried by  $\tau_n$ , for some  $n$ -tower of derived train tracks  $\tau = \tau_0 \supset \tau_1 \supset \dots \supset \tau_n$  with  $n \geq 2$ . Then the distance of the given Heegaard splitting satisfies:*

$$d(V, W) \geq n$$

The technical terms of this theorem are explained in detail in Sections 2, 3 and 4. A innovative feature of this paper are *fat* train tracks (compare Definitions 3.1 and 3.3), which allow a smooth transition from disk systems in a handlebody to the train track technology needed to obtain lower distance bounds in the curve complex.

This train track technology, presented below in Section 2, is based on our concept of a *derived* train track, which is a combinatorial version of Masur-Minsky’s “Basic observation” ([11], §3.1) that turns out to be very handy in our context.

Special effort was made to give all proofs of this paper from basic notions.

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## 2. TRAIN TRACKS AND DISTANCE IN THE CURVE COMPLEX

Recall that for an orientable connected surface  $\Sigma$  of genus  $g \geq 2$  the curve complex  $\mathcal{C}(\Sigma)$  is defined as follows:

- (1) The set of vertices  $\mathcal{C}^0(\Sigma)$  is the set of isotopy classes of simple closed curves on  $\Sigma$ .
- (2) An  $n$ -simplex is a collection  $\{v_0, \dots, v_n\}$  of vertices which have mutually disjoint representative curves.

On the 1-skeleton of  $\mathcal{C}^1(\Sigma)$  we define a metric  $d_{\mathcal{C}}(\cdot, \cdot)$  by declaring the length of every edge to be 1. For the purpose of this paper it will suffice to consider only  $\mathcal{C}^1(\Sigma)$ .

The curve complex  $\mathcal{C}(\Sigma)$  was invented by Harvey [5] and has been the object of intense research in recent years (see [1], [6], [8], [9], [11], [12], [13], [14], just to mention a few from a much longer list). Its

importance is highlighted by the fact that this complex, on which the mapping class group  $MCG(\Sigma)$  acts naturally (but not properly, as  $\mathcal{C}(\Sigma)$  is not locally compact), is indeed a  $\delta$ -hyperbolic space in the sense of Gromov (see [11]). For background and more information on the curve complex see [14].

We now briefly review *train tracks* from a somewhat non-standard perspective which will be used in the next two Sections:

A *train track*  $\tau$  in a closed surface  $\Sigma$  is a closed subsurface with a *singular  $I$ -fibration*. By this we mean that the interior of  $\tau$  is fibered by open arcs (each homeomorphic to the interior of the unit interval  $I$ ) and that the fibration extends to a fibration of the closed surface  $\tau$  by properly embedded closed arcs, except for finitely many *singular points on  $\partial\tau$* , where precisely two fibers meet. We call these fibers *singular fibers*. We admit the case that a fiber is *doubly singular*, i.e., both of its endpoints are singular.

Two singular fibers are *adjacent* if they share a singular point as a common endpoint. A maximal connected union of singular or doubly singular  $I$ -fibers is called an *exceptional fiber*. It consists of a sequence of adjacent fibres, and hence it is either homeomorphic to a closed interval, or to a simple closed curve on  $\Sigma$ . In the latter case it will be called a *cyclic exceptional fiber*. We explicitly admit this second case, although we are aware of the fact that in the classical train track literature this case is sometimes suppressed.

We picture a singular point  $P \in \partial\tau$  in such a way that two arcs from  $\partial\tau$  which intersect in  $P$  converge towards  $P$  from the “same direction”, thus giving rise to a *cuspid point* on the boundary of the corresponding complementary component of  $\tau$  in  $\Sigma$ .

We define the *type* of a complementary component  $\Delta$  of  $\tau$  in  $\Sigma$  as given by the genus of  $\Delta$  and the number of cusp points on its boundary. If  $\Delta$  is simply connected, we speak of an  $n$ -gon if there are precisely  $n$  cusp points on  $\partial\Delta$ . For example, if  $\partial\Delta$  contains precisely three cusp points, we say that  $\Delta$  is a *triangle*. An arc of  $\partial\Delta$  which joins two adjacent cusp points is called a *side* of  $\Delta$ . If all complementary regions of the train track  $\tau$  are triangles we say that  $\tau$  is *maximal*.

**Remark 2.1.** Contracting every  $I$ -fiber to a point defines a deformation of  $\tau$  to a graph  $\Gamma_\tau$ . The exceptional fibers will be deformed to the vertices of  $\Gamma_\tau$ . The  $I$ -fiber structure on  $\tau$  defines a gate structure on  $\Gamma_\tau$ , which is usually visualized by giving  $\Gamma_\tau$  the structure of a branched 1-manifold.

The deformation  $\tau \rightarrow \Gamma_\tau$  is a homotopy equivalence only if there are no cyclic exceptional fibers. Also,  $\Gamma_\tau$  can be embedded as a retract into  $\tau$  only if every exceptional fiber is *simple*, i.e., it consists of exactly two singular  $I$ -fibers. In this case the corresponding vertex of  $\Gamma_\tau$  can be mapped to the unique cusp point  $P$  contained in the exceptional fiber. Notice that even in this case the subsurface  $\tau \subset \Sigma$  fails to be a regular neighborhood of the embedded graph  $\Gamma_\tau \subset \Sigma$ , exactly at the singular points of  $\partial\tau$ .

**Remark 2.2.** If  $\tau$  is connected and has at least one exceptional fiber, then any closed curve  $c \subset \tau$  which does not meet any of the exceptional fibers of  $\tau$  is contractible in  $\tau$ . This follows directly from the fact that for the above homotopy equivalence  $\tau \rightarrow \Gamma_\tau$  the image of  $c$  is a loop in the graph  $\Gamma_\tau$  that is disjoint from all vertices of  $\tau$ . Note that  $\Gamma_\tau$  is connected and topologically different from  $S^1$ , by the assumption on  $\tau$ .

An arc or a closed curve in  $\Sigma$  is *carried* by a train track  $\tau \subset \Sigma$  if it is contained in  $\tau$  and is always transverse to the  $I$ -fibers in  $\tau$ . A simple closed curve *can be carried* by  $\tau$  if it can be isotoped in  $\Sigma$  to a curve that is carried by  $\tau$ . The two notions are not always being kept strictly apart, as most of the time one is interested in curves only up to isotopy on  $\Sigma$ .

Two simple arcs or curves carried by  $\tau$  are *parallel* if they intersect the same  $I$ -fibers, and these intersections occur on the two arcs (or curves) in precisely the same order.

**Definition 2.3.** Let  $\tau$  be a train track in a surface  $\Sigma$ . An arc or a closed curve on  $\Sigma$  which is carried by  $\tau$  is said to *cover*  $\tau$  if it meets every  $I$ -fiber of  $\tau$ .

The statement of Lemma 2.5 below has been first observed in [11], Section 3.1, “A basic observation”. We include here a detailed proof of this and the preceding Lemma 2.4, as we will return later, in Section 3, to the same argument, which will be generalized (see the proof of Lemma 3.9).

**Lemma 2.4.** *Let  $\tau \subset \Sigma$  be a train track, and let  $\mathcal{D}$  be a finite collection of pairwise disjoint simple closed curves which together cover  $\tau$ . Then for every complementary component  $\Delta$  of  $\tau$  in  $\Sigma$  and every side  $\delta$  of  $\Delta$  there is an arc  $\alpha$  on some curve of  $\mathcal{D}$  that runs parallel along all of  $\delta$ .*

*Proof.* Let  $\sigma$  be an  $I$ -fiber of  $\tau$  that meets  $\delta$  in a point  $P \in \partial\sigma \cap \delta$ , and let  $Q$  be the point of  $\sigma \cap \mathcal{D}$  located on  $\sigma$  closest to  $P$ . Let  $\alpha$  be

the maximal arc of  $\mathcal{D}$  that contains  $Q$  and runs parallel to a subarc of  $\delta$ . If  $\alpha$  does not run parallel along all of  $\delta$ , it branches off at some singular  $I$ -fiber  $\sigma'$  which has a (non-singular) endpoint on  $\delta$ . Since  $\mathcal{D}$  covers  $\tau$ , there must be another arc  $\beta$  of  $\mathcal{D}$  that meets  $\sigma'$ . But then the elongation of  $\beta$ , in the direction of  $\alpha$  towards  $\sigma$ , will proceed between  $\alpha$  and  $\delta$ , and thus give an intersection point with  $\sigma$  which lies between  $P$  and  $Q$ , contradicting the choice of  $Q$ . Thus  $\alpha$  cannot branch off from  $\delta$  and hence runs parallel along  $\delta$  all the way.  $\square$

**Lemma 2.5.** *Let  $\tau$  be a maximal train track on a surface  $\Sigma$ . Let  $\mathcal{D}$  be a finite collection of pairwise disjoint simple closed curves which together cover  $\tau$ , and let  $D$  be an essential simple closed curve disjoint from  $\mathcal{D}$ . Then  $D$  can be carried by  $\tau$ .*

*Proof.* We can isotope  $D$  off all complementary components of  $\tau$  so that, after the isotopy, the image of  $D$  in  $\tau$  will still be disjoint from  $\mathcal{D}$ . This can be done at the expense of accepting that at finitely many singular points the curve  $D$  is not transverse to the  $I$ -fibering. We can now shorten  $D$  at those singularities and thus eventually eliminate all of them except those that occur in the neighborhood of a cusp point  $P$  of some complementary component  $\Delta$ . In this case there are two arcs  $\alpha$  and  $\alpha'$  on  $D$  adjacent to such a singularity  $X \in D$ , which run parallel to initial segments of the two sides  $\delta$  and  $\delta'$  of  $\Delta$  that are adjacent to  $P$ . Choose such a singularity  $X$  and two such arcs  $\alpha$  and  $\alpha'$  which are closest to  $\delta$  and  $\delta'$  respectively. Notice that it is impossible that there is a curve from  $\mathcal{D}$  that passes between  $\alpha$  and  $\delta$  or between  $\alpha'$  and  $\delta'$ , as the singularity  $X$  lies on both,  $\alpha$  and  $\alpha'$ : Since  $D$  is disjoint from  $\mathcal{D}$  and the latter is carried by  $\tau$ ,  $X$  cannot be separated from the cusp point  $P$  by curves from  $\mathcal{D}$ .

By Lemma 2.4, applied to  $\mathcal{D}$ , the arcs  $\alpha$  and  $\alpha'$  can be elongated to arcs  $\hat{\alpha}$  and  $\hat{\alpha}'$  of  $D$  that are parallel to the entire sides  $\delta$  and  $\delta'$  respectively. But then we can isotope the union  $\hat{\alpha} \cup \hat{\alpha}'$  across  $\Delta$  to get an arc  $\alpha''$  that runs parallel to the third side  $\delta''$  of  $\Delta$ , thus eliminating the singularity  $X \in D$ . Finitely many successive such isotopies bring  $D$  into a position where it is contained in  $\tau$  and has no singularities with respect to the  $I$ -fibering: It is thus carried by  $\tau$ .  $\square$

**Remark 2.6.** Assume that  $\Sigma$  is endowed with some hyperbolic structure. Recall that a (geodesic) lamination  $\mathcal{L}$  is a non-empty closed subset of  $\Sigma$  which is a disjoint union of simple geodesics called *leaves* (see, e.g.,

Casson-Bleiler [2]). We say that  $\mathcal{L}$  is *carried* by a train track  $\tau$  if each of its leaves is carried by  $\tau$ . An example, which will do for most of the rest of this paper and doesn't need the hyperbolic structure on  $\Sigma$ , is simply to take for  $\mathcal{L}$  a (typically rather long) essential simple closed curve, or a finite pairwise disjoint collection of such.

Given a train track  $\tau \subset \Sigma$  which carries a lamination  $\mathcal{L}$  we can obtain a new train track as follows:

The train track  $\tau$  can be *split* by moving any of the cusp points  $P$  (now called a *zipper*), which is located on the boundary of a complementary component  $\Delta$  of  $\tau$ , in an “outward” direction with respect to  $\Delta$ . In other words,  $P$  is moved into the interior of  $\tau$ . The zipper  $P$  will move along an *unzipping path*, which is embedded in the interior of  $\tau \setminus \mathcal{L}$ . Furthermore the unzipping path is always transverse to the  $I$ -fibers. Two unzipping paths are not allowed to cross each other. In case two zippers meet the same connected component of an  $I$ -fiber in  $\tau \setminus \mathcal{L}$  from different directions, they have to join up, thus changing the topology of the train track and of its complementary components. A situation like this is called a *collision*. In case of a collision the unzipping procedure stops. An unzipping path which covers  $\tau$  is called *complete*.

**Definition 2.7.** We say that  $\tau$  can be derived with respect to  $\mathcal{L}$  if we can successively (or simultaneously, it does not make any difference) unzip every zipper along a complete unzipping path, without ever running into a collision. The train track  $\tau'$  obtained by unzipping along paths which are shortest possible, complete unzipping paths is said to be *derived from  $\tau$  with respect to  $\mathcal{L}$* , or simply, if the context is clear, *derived from  $\tau$* .

**Remark 2.8.** If the train track  $\tau'$  is derived from a train track  $\tau$  then every complementary component  $\Delta$  of  $\tau'$  is of the same type as the complementary component of  $\tau$  which is contained in  $\Delta$ . This follows directly from Definition 2.7, since during the deriving process the unzipping paths never run into collisions. In particular, if  $\tau$  is maximal, then so is  $\tau'$ .

**Lemma 2.9.** *Given a surface  $\Sigma$  and maximal train tracks  $\tau, \tau' \subset \Sigma$  so that  $\tau'$  is derived from  $\tau$ , let  $D$  be a simple closed essential curve carried by  $\tau'$ . Then  $D$  covers  $\tau$ .*

*Proof.* Since  $D$  is essential, it has to meet some singular  $I$ -fiber  $\sigma$  of  $\tau'$ , by Remark 2.2. Let  $P \in \partial\sigma$  be a singular point:  $P$  is also a cusp point on a complementary component  $\Delta$  of  $\tau'$ . Let  $\delta$  be the side of  $\Delta$  adjacent to  $P$ , on the side of  $\sigma$ . Let  $\alpha$  be an arc on  $D$  which starts at  $\sigma$  and runs parallel to an initial segment of  $\delta$ .

By the assumption that  $\tau'$  is derived from  $\tau$ , the cusp point  $P$  is the endpoint of some unzipping path  $\zeta \subset \Delta$  which runs parallel in  $\tau$  to an initial segment of  $\delta$ , starting at  $P$ . There are two cases:

- (i) The path  $\alpha$  runs parallel in  $\tau$  along all of the unzipping path  $\zeta$ .
- (ii) There is another singular  $I$ -fiber of  $\tau'$  which has its non-singular endpoint  $P'$  on  $\delta$ , while its singular endpoint is a cusp point  $Q$  of another complementary component  $\Delta'$  of  $\tau'$ . The complementary component  $\Delta'$  has two sides  $\epsilon$  and  $\delta'$  adjacent to  $Q$ , such that the beginning of  $\delta'$  runs parallel to  $\delta$  while  $\alpha$  runs along  $\epsilon$ . (i.e.,  $\alpha$  “splits” off  $\delta$  at  $Q$ .)

Now either  $\epsilon$  and  $\delta'$  (and thus  $\delta$ ) run parallel on  $\tau$  until the end of the unzipping path  $\zeta$  is reached, or else the unzipping path  $\zeta'$  for  $\Delta'$  which separates  $\epsilon$  from  $\delta'$  is strictly shorter than  $\zeta$  (where “length” is measured in terms of the transverse  $I$ -fibers that are being intersected by  $\zeta$  and by  $\zeta'$ ). By this we mean that there is a proper subpath of  $\zeta$  which is parallel on  $\tau$  to  $\zeta'$ . We can rule out this case by assuming at the beginning of this proof that the arc  $\alpha$  on  $D$  starts at a shortest possible unzipping path. Thus an initial segment  $\epsilon'$  of  $\epsilon$  has to run parallel in  $\tau$  to some unzipping path  $\zeta$ . If  $\alpha$  branches again off  $\epsilon$  before the end of  $\epsilon'$  is reached, we repeat the exact same argument just given.

This shows that  $\alpha$  has to run parallel to some unzipping path for  $\tau'$ , and hence it covers  $\tau$ , by the properties of a derived train track as given in Definition 2.7.

□

**Remark 2.10.** Since Lemma 2.9 is the crucial place in this paper where we use the properties of a derived train track, it is natural to ask whether a weaker condition may still allow the same conclusion. Indeed, one can work with a train track  $\tau'$  that is *weakly derived* from a given train track  $\tau$ , by which we mean that  $\tau'$  is obtained by unzipping  $\tau$ , and that every circuit carried by  $\tau'$  covers  $\tau$ . Here a *circuit* means a loop which embeds under the natural map  $\tau \rightarrow \Gamma_\tau$ .

As there are only finitely many circuits in any train track, this definition is still practical to check, and the results of this paper generalize to this weaker (but more cumbersome) notion.

**Definition 2.11.** A collection of train tracks  $\tau_0 \supset \tau_1 \supset \dots \supset \tau_n$  will be called an *n-tower of derived train tracks in  $\Sigma$*  if there is some lamination  $\mathcal{L} \subset \Sigma$  so that  $\tau_i$  is derived from  $\tau_{i-1}$  with respect to  $\mathcal{L}$  for all  $i = 1, \dots, n$ .

Sequences of nested train tracks, as given in the previous definition, occur already in [11], §3.1, where they are used to derive lower bounds for the distance in the curve complex. Indeed, the following proposition is a variant of their basic observation, presented here in the slightly more combinatorial setting which we prefer in this paper, for the benefit of our use in the later sections in the context of Heegaard splittings of 3-manifolds.

**Proposition 2.12.** *Let  $\tau_0 \supset \tau_1 \supset \dots \supset \tau_n$  be an n-tower of derived train tracks in  $\Sigma$ . Assume that  $\tau_0$  (and hence of any of the  $\tau_i$ ) is maximal. Let  $D$  be a simple closed curve carried by  $\tau_n$ . Then any essential simple closed curve  $D'$  which satisfies  $d_C(D, D') \leq n$  can be carried by  $\tau_0$ .*

*Proof.* Any curve carried by some of the  $\tau_{i+1}$  covers  $\tau_i$ , by Lemma 2.9. A second curve disjoint from the first can be then carried by  $\tau_i$ , by Lemma 2.5. Hence we see recursively that for any family of essential simple closed curves  $D = D_0, D_1, \dots, D_n = D'$ , where any subsequent pair  $D_i, D_{i+1}$  is disjoint, each  $D_i$  can be carried by  $\tau_{n-i}$  and covers  $\tau_{n-i-1}$ . In particular,  $D'$  can be carried by  $\tau_0$ . □

**Remark 2.13.** The last proposition can be reformulated to say that any simple closed essential curve  $E$  that cannot be carried by  $\tau_0$  satisfies

$$d_C(D, E) \geq n + 1.$$

Thus, in order to apply this result in practice, we need a convenient criterion to ensure that a given curve  $E$  cannot be carried by the train track  $\tau_0$ . For this purpose we introduce, in the next section, the notion of a fat train track.

### 3. FAT TRAIN TRACKS

In this section we will investigate a special class of train tracks that are very useful and natural in the context of this paper.



**Definition 3.1.** A train track  $\tau \subset \Sigma$  is called *fat* if all of its exceptional fibers are cyclic. We denote by  $\mathcal{E}_\tau$  the collection of simple closed curves on  $\Sigma$  given by the exceptional fibers of  $\tau$ .

A system  $\mathcal{E}$  of essential simple closed curves on  $\Sigma$  is called a *complete decomposing system* if every complementary component of  $\mathcal{E}$  in  $\Sigma$  is a *pair-of-pants*, i.e., a sphere with three open disks removed.

If a complete decomposing system is given, then we will assume that any essential simple closed curves  $D$  on  $\Sigma$  that is considered shall be transverse to  $\mathcal{E}$ . Similarly, any simple arc  $\alpha$  must be transverse to  $\mathcal{E}$ , and we will only consider arcs that have their endpoints on  $\mathcal{E}$ . As before, such  $D$  or  $\alpha$  are considered only up to isotopy: However, we will only allow isotopies of the pair  $(\Sigma, \mathcal{E})$ .

A curve  $D$  or such an arc  $\alpha$  is called *tight* with respect to  $\mathcal{E}$  if the number of intersection points with  $\mathcal{E}$  cannot be strictly decreased by an isotopy of  $D$  or  $\alpha$ .

**Definition 3.2.** Let  $P \subset \Sigma$  be a pair-of-pants.

- (a) A simple arc in  $P$  which has its two endpoints on different components of  $\partial P$  will be called a *seam*.
- (b) A simple arc in  $P$  which has both endpoints on the same component of  $\partial P$ , and is not  $\partial$ -parallel, will be called a *wave*.
- (c) An essential simple closed curve  $D \subset \Sigma$  has a *wave* with respect to a complete decomposing system  $\mathcal{E} \subset \Sigma$  if  $D$  is tight with respect to  $\mathcal{E}$  and if it contains a subarc that is a wave in a complementary component  $P_i$  of  $\mathcal{E}$  in  $\Sigma$ .
- (d) An essential simple closed curve  $D \subset \Sigma$  has a *wave* with respect to a fat train track  $\tau$  if  $D$  has a wave with respect to  $\mathcal{E}_\tau$ , or if  $D$  is isotopic to some  $E_k \in \mathcal{E}_\tau$ .

Let  $P \subset \Sigma$  be a pair-of-pants. Place a *vertex* on each of its three boundary components and connect any two of them by disjoint simple arcs. The three arcs together bound a subsurface  $\Delta_1 \subset P$ , called a *triangle*. Repeat this operation on  $P \setminus \Delta_1$  to obtain a second such triangle denoted by  $\Delta_2$ . Note that  $P \setminus \text{interior}(\Delta_1 \cup \Delta_2)$  is a collection of three *rectangles*  $R_1, R_2$  and  $R_3$ , since the boundary of each of  $R_j$  (for  $j = 1, 2, 3$ ) is composed of four arcs separated by four of the six vertices introduced above. Define an  $I$ -fibration on each of the rectangles  $R_j$  by filling  $R_j$  with arcs parallel to the two arcs from  $\partial R_j \cap \partial P$ .

Let  $\mathcal{E}$  be a complete decomposing system on  $\Sigma$ , and consider the collection of complementary pair-of-pants  $P_i \subset \Sigma$ , i.e.,  $\Sigma = \bigcup_{i=1}^{2g-2} P_i$ .

Put the structure as above on each of pair-of-pants  $P_i$ , in such a way that on each curve  $E_k$ , one has placed four distinct vertices: two vertices from each of the two pair-of-pants adjacent to  $E_k$ .

The  $I$ -fibrations on any of the rectangles in each of the  $P_i$  join up to define a train track structure  $\tau$  on  $\Sigma \setminus \text{interior}(\Delta)$ , where  $\Delta$  is the union of all the triangles  $\Delta_j$  in any of the  $P_i$ . For this train track the set of exceptional fibers  $\mathcal{E}_\tau$  is exactly the complete decomposing system  $\mathcal{E}$  we started out with, and they are all cyclic, i.e.,  $\tau$  is a fat train track. Furthermore, the train track  $\tau$  has no complementary region other than the two triangles in each of the pair-of-pants  $P_i$ , and those are indeed triangles in the meaning of Section 2. Finally, consider any arc  $\alpha$  that intersects  $\mathcal{E}$  precisely in its endpoints: we observe that  $\alpha$  can be carried by  $\tau$  if and only if it is a seam: In particular, no wave (with respect to  $\mathcal{E}_\tau$ ) is carried by the train track  $\tau$ .

This construction gives rise to the following:

**Definition 3.3.** A fat train track  $\tau \subset \Sigma$  is called *complete* if the following conditions are satisfied:

- (1) The collection  $\mathcal{E}_\tau$  of exceptional fibers of  $\tau$  is a complete decomposing system on  $\Sigma$ .
- (2) Each pair-of-pants  $P_i$  complementary to the system  $\mathcal{E}_\tau$  contains two triangles as complementary components of  $\tau$  in  $P_i$ .
- (3) The train track  $\tau$  only carries seams, but no waves, with respect to the complete decomposing system  $\mathcal{E}_\tau$ .

**Remark 3.4.** Given a complete decomposing system  $\mathcal{E} \subset \Sigma$ , there are only finitely many complete fat train tracks  $\tau$  with  $\mathcal{E}_\tau = \mathcal{E}$ , up to orientation preserving homeomorphisms of  $\Sigma$  which leave invariant every curve of  $\mathcal{E}$  and every complementary pair-of-pants.

To be precise, for each curve  $E \in \mathcal{E}$  there are six possible configurations for the vertices of the adjacent four triangles, up to a homeomorphism as above. Hence we have  $6^{3g-3}$  possible configurations for a complete fat train track, up to such homeomorphisms.

By definition every complete fat train track is maximal.

**Lemma 3.5.** *Let  $\mathcal{E} \subset \Sigma$  be a complete decomposing system. Any essential simple closed curve  $D \subset \Sigma$  which does not have a wave with respect to  $\mathcal{E}$ , and is not parallel to any  $E_k \in \mathcal{E}$ , is carried by some complete fat train track  $\tau$  with exceptional fibers  $\mathcal{E}_\tau = \mathcal{E}$ .*

*The same is true for any system of pairwise disjoint essential simple closed curves which satisfy the same conditions as  $D$ .*

*Proof.* If  $D$  is not parallel to any  $E_k \in \mathcal{E}$  and has no waves with respect to  $\mathcal{E}$ , then for each of the pair-of-pants  $P_i$  complementary to  $\mathcal{E}$  in  $\Sigma$  the connected components of  $D \cap P$  must all be seams. Since any two seams which join the same two boundary components of  $P_i$  are necessarily parallel (because of the simple topology of a pair-of-pants), it follows that the parallelity classes of such seams can be grouped together and isotoped so that they are contained in a transversely  $I$ -fibered rectangle in  $P_i$  as introduced above. After introducing, if necessary, some additional empty  $I$ -fibered rectangles, the complement of the rectangles in each  $P_i$  will consist precisely of two triangles. Thus the union of all  $I$ -fibered rectangles defines a complete fat train track that carries  $D$  and satisfies  $\mathcal{E}_\tau = \mathcal{E}$ .  $\square$

The same type of argument can be used to obtain the following statement:

**Remark 3.6.** Let  $\mathcal{E}$  be a complete decomposing system on the surface  $\Sigma$ , and let  $D$  be an essential simple closed curve (or a system of such curves) on  $\Sigma$  that is tight with respect to  $\mathcal{E}$ . We say that  $D$  *fills* a pair-of-pants  $P$  complementary to  $\mathcal{E}$ , if  $D \cap P$  is the disjoint union of precisely 3 distinct isotopy classes of intersection arcs. As before we consider here isotopy of the pair  $(P, \partial P)$ . Then the following three statements are equivalent:

- (1) The curve  $D$  fills every pair-of-pants complementary to  $\mathcal{E}$ , and none of the intersection arcs is a wave.
- (2) There exists a unique complete fat train track  $\tau$  with exceptional fibers  $\mathcal{E}_\tau = \mathcal{E}$  that carries  $D$ .
- (3) There exists some complete fat train track  $\tau$  with exceptional fibers  $\mathcal{E}_\tau = \mathcal{E}$  that is covered by  $D$ .

**Lemma 3.7.** *Let  $\tau$  be a complete fat train track, and let  $\tau'$  be derived from  $\tau$ . Let  $D$  be an essential simple closed curve that is carried by  $\tau'$ , and let  $E$  be an essential simple closed curve that has a wave with respect to  $\tau$ . Then*

$$d_{\mathcal{C}}(D, E) \geq 2$$

*Proof.* We apply Lemma 2.9 to deduce that the curve  $D$  covers  $\tau$ . But any curve that covers  $\tau$  must intersect any of the  $E_j \in \mathcal{E}_\tau$ , and also any wave in any of the complementary components  $P_i$  of the complete decomposing system  $\mathcal{E}_\tau$  in  $\Sigma$ . Thus  $D$  intersects the curve  $E$  and hence it is of distance at least two from it:  $d_{\mathcal{C}}(D, E) \geq 2$ .  $\square$

**Corollary 3.8.** *Let  $\tau_0 \supset \tau_1 \supset \dots \supset \tau_n$ ,  $n \geq 1$ , be an  $n$ -tower of derived train tracks in  $\Sigma$ . Assume that  $\tau_0$  is a complete fat train track. Let  $D$  be an essential simple closed curve carried by  $\tau_n$ , and let  $E$  be an essential simple closed curve which has a wave with respect to  $\tau_0$ . Then one has:*

$$d_C(D, E) \geq n + 1$$

*Proof.* By Proposition 2.12 any curve  $D'$  of distance at most  $n - 1$  from  $D$  is carried by  $\tau_1$ . Hence we can apply Lemma 3.7 to deduce that  $D'$  has distance greater or equal to 2 from  $E$ . Thus  $d_C(E, D) \geq n + 1$ .  $\square$

We now prove a useful analogue of Lemma 2.5, for simple arcs rather than simple closed curves. In order to “fix” the endpoints of such an arc  $\alpha$  we require here and in the next sections that  $\partial\alpha$  is contained in a complete decomposing system  $\mathcal{E}$ , and that  $\alpha$  is tight with respect to  $\mathcal{E}$ . The system  $\mathcal{E}$ , in our context, is given as set of singular fibers  $\mathcal{E} = \mathcal{E}_\tau$  of some complete fat train track  $\tau$ . Thus the role of this train track  $\tau$  is in some sense that of a “coordinate system”, while the train track  $\tau'$ , the one we are really interested in, is a much finer and longer train track than  $\tau$ : we only require that  $\tau'$  is derived from  $\tau$ .

**Lemma 3.9.** *Let  $\tau$  be a complete fat train track on a surface  $\Sigma$ , and let  $\tau'$  be a maximal train track derived from  $\tau$ . Let  $\beta$  be an arc with endpoints on  $\mathcal{E}_\tau$  which covers  $\tau'$ . Let  $D$  be an essential simple closed curve which is tight with respect to  $\mathcal{E}_\tau$  and contains  $\beta$  as subarc. Then  $D$  can be carried by  $\tau'$ , and in fact covers  $\tau'$ .*

*Proof.* We first observe that the analogue of the statement of Lemma 2.4 holds, with  $D$  replaced by  $\beta$ : the proof given in Section 2 applies word by word to this generalization.

We now proceed precisely as in the proof of Lemma 2.5: We first use an isotopy that fixes  $\beta$  and all intersection points of  $D$  with  $\mathcal{E}_\tau$  to isotope  $D$  off all complementary components of  $\tau'$ , while making sure that it stays tight with respect to  $\mathcal{E}_\tau$ , and also that it stays simple. We then decrease the number of singularities successively, as in the proof of Lemma 2.5 (using the above mentioned analogue of Lemma 2.4). At the end state of this procedure there are no singularities left, and  $D$  has been isotoped into a position where it is carried by  $\tau'$ . But  $D$  contains  $\beta$  as subarc, which covers  $\tau'$  by assumption. Thus  $D$  covers  $\tau'$ .  $\square$

Let  $k$  be an essential simple closed curve which is tight with respect to the complete decomposing system  $\mathcal{E}$  on  $\Sigma$ . The number  $|k \cap \mathcal{E}|$  of intersection points of  $k$  with  $\mathcal{E}$  is called the  $\mathcal{E}$ -length of  $k$  and is denoted by  $|k|_{\mathcal{E}}$ . The same definition and notation will be used for a simple arc  $\alpha$  instead of  $k$ , where as before we require  $\partial\alpha \in \mathcal{E}$ .

Recall that two tight simple arcs  $\alpha, \alpha'$  on  $\Sigma$  are called parallel (with respect to  $\mathcal{E}$ ) if, after orienting them properly, they intersect  $\mathcal{E}$  in precisely the same sequence of curves  $E_{i_1}, \dots, E_{i_k}$ , and if the intersections occur from the same direction. This is equivalent to saying that the arcs are isotopic by an isotopy of the pair  $(\Sigma, \mathcal{E})$ .

Let  $c$  and  $k$  be distinct simple closed curves on  $\Sigma$  that are tight with respect to  $\mathcal{E}$ , and let  $P \in c \cap k$  be an intersection point. We denote by  $|P|_{\mathcal{E}}$  the  $\mathcal{E}$ -length of any of two maximal arcs on  $k$  and on  $c$ , which are parallel and which both contain  $P$ . In the last sentence, the terminology “arc on a closed curve” needs to be specified: Such an arc is not necessarily a subarc, but it can also be an arc that winds several times around the closed curve, thus being immersed but not embedded in the curve.

We define the *twisting number of  $k$  along  $c$  at  $P$*  to be the quotient:

$$tw_P(k, c) = \frac{|P|_{\mathcal{E}}}{|c|_{\mathcal{E}}}$$

**Lemma 3.10.** *Let  $\mathcal{E}$  be a complete decomposing system on  $\Sigma$ , and let  $k$  and  $c$  be essential simple curves on  $\Sigma$  that are tight with respect to  $\mathcal{E}$ . Assume that  $k$  and  $c$  only intersect essentially, and let  $P \in c \cap k$  be such an essential intersection point.*

- (a) *The Dehn twist  $\delta_c$  at  $c$  effects the twisting number as follows:*

$$tw_P(\delta_c^m(k), c) = |tw_P(k, c) + \varepsilon m|,$$

*where  $\varepsilon = \pm 1$  is independent of  $m$ .*

- (b) *Assume  $|tw_P(k, c)| > 1$  and assume furthermore that  $c$  covers some train track  $\tau'$  that is derived from a complete fat train track  $\tau$  with  $\mathcal{E}_{\tau} = \mathcal{E}$ . Then  $k$  covers  $\tau'$  as well. In particular,  $k$  does not have waves with respect to  $\mathcal{E}$ .*

*Proof.* Statement (a) of this lemma is a direct consequence of the above definitions. Here the role of the constant  $\varepsilon$  is geometrically explained as follows: The Dehn twist  $\delta_k$  is a “right-handed” twist. On the other hand, before applying the Dehn twist, the curve  $k$  may already, at the intersection point  $P$ , wind around  $c$ , either in the “right hand” sense, or in the “left hand” one. In the first case one has to set  $\varepsilon = 1$ , while in the second case one has  $\varepsilon = -1$ .

To prove (b), we first note that  $k$  is not assumed to be carried by  $\tau'$ . However, it suffices to apply Lemma 3.9, where  $\beta$  is the maximal subarc on  $k$  which contains  $P$  and is parallel to an arc on  $c$  that also contains  $P$ .

□

**Remark 3.11.** (a) Since in Lemma 3.10  $k$  and  $c$  are both simple, it follows directly from the definitions that for any two intersection points  $P, Q \in c \cap k$  the twisting numbers  $tw_P(k, c)$  and  $tw_Q(k, c)$  cannot differ by more than 1. The same is true for  $tw_Q(k', c)$  rather than  $tw_Q(k, c)$ , where  $k'$  is any second essential simple closed curve disjoint from  $k$  that is tight with respect to  $\mathcal{E}$  and intersects  $c$  essentially in  $Q$ . (b) The above definitions as well as the statement of Lemma 3.10 stay valid if the curve  $k$  is replaced by a lamination  $\mathcal{L} \subset \Sigma$ , where every leaf of  $\mathcal{L}$  is supposed to be tight with respect to  $\mathcal{E}$ .

**Proposition 3.12.** *Let  $\mathcal{D}$  and  $\mathcal{E}$  be complete decomposing systems on  $\Sigma$ , and let  $k$  be an essential simple curve on  $\Sigma$  that is tight with respect to both,  $\mathcal{D}$  and  $\mathcal{E}$ . Assume that the curve  $k$  fills every pair-of-pants complementary to  $\mathcal{D}$  or to  $\mathcal{E}$ , and that, furthermore,  $k$  contains no wave with respect to either  $\mathcal{D}$  or  $\mathcal{E}$ .*

*Then there exists an integer  $m_0$  such that for every  $m \in \mathbb{Z} \setminus \{m_0, m_0 + 1, m_0 + 2, m_0 + 3\}$  one has:*

- (a) *The complete decomposing system  $\mathcal{D}^m = \delta_k^m(\mathcal{D})$ , obtained from  $\mathcal{D}$  via  $m$ -fold Dehn twist on  $k$ , has the property that  $\mathcal{D}^m$  and  $\mathcal{E}$  do not have waves with respect to each other.*
- (b) *If  $k$  covers a maximal train track  $\tau'$  that is derived from some complete fat train track  $\tau$  with exceptional fibers  $\mathcal{E}_\tau = \mathcal{E}$ , then  $\mathcal{D}^m$  as well covers  $\tau'$  (and hence is carried by  $\tau'$ ).*

*Proof.* (I.) In this first part of the proof we only consider the system  $\mathcal{D}$ , and we want to investigate which values  $m \in \mathbb{Z}$  have the property (actually a slightly stronger one) that the system  $\mathcal{D}^m$  does not contain waves with respect to  $\mathcal{E}$ .

We first consider an essential intersection point  $P$  of  $k$  with some curve  $D_i$  of the system  $\mathcal{D}$ . We apply statement (a) of Lemma 3.10 to see that one of the following two cases occurs, according to whether  $tw_P(D_i, k)$  is an integer (case (1)) or not (case (2)):

- (1) There exists an integer  $m_1 \in \mathbb{Z}$  such that

$$tw_P(\delta_k^m(D_i), k) = |m - m_1|$$

for all  $m \in \mathbb{Z}$ .

- (2) There exists an integer  $m_1 \in \mathbb{Z}$  and real numbers  $c_-, c_+ \in (0, 1)$  such that:
- (a)  $tw_P(\delta_k^{m_1}(D_i), k) = c_-$
  - (b)  $tw_P(\delta_k^{m_1+1}(D_i), k) = c_+$
  - (c)  $tw_P(\delta_k^m(D_i), k) = |m - m_1 - 1 + c_+|$  if  $m \geq m_1 + 1$
  - (d)  $tw_P(\delta_k^m(D_i), k) = |-m + m_1 + c_-|$  if  $m \leq m_1$ .

A second curve  $D_j \in \mathcal{D}$  and any intersection point  $Q \in D_j \cap k$  satisfies the same statement, and furthermore we observed in Remark 3.11 (a) that  $tw_P(\delta_k^m(D_i), k)$  and  $tw_Q(\delta_k^m(D_j), k)$  can not differ by more than 1.

We now want to define the *exceptional  $\mathcal{D}$ -values*  $m_-, \dots, m_+$  in  $\mathbb{Z}$  by defining their complement in  $\mathbb{Z}$ :

An integer  $m$  is a *non-exceptional  $\mathcal{D}$ -value* if at every intersection point  $P$  of  $\delta_k^m(D_i)$  with  $k$ , for any  $D_i \in \mathcal{D}$ , one has

$$tw_P(\delta_k^m(D_i), k) \geq 1,$$

and for at least one such  $P$  and  $D_i$  the inequality is strict.

By checking for each  $D_i \in \mathcal{D}$  the two possible cases (1) and (2) above, we deduce that there are at most three adjacent *exceptional  $\mathcal{D}$ -values*  $m_-, \dots, m_+$  in  $\mathbb{Z}$ . Furthermore, we deduce from the above definition of the non-exceptional values, that for every integer  $m \geq m_+ + 1$  the sum  $S_m$  of all twisting numbers, over all intersection points of any  $\delta_k^m(D_i) \in \delta_k^m(\mathcal{D})$  with  $k$ , satisfies:

$$S_m = S_{m-1} + |\mathcal{D} \cap k|$$

Similarly, for any integer  $m \leq m_- - 1$  we obtain:

$$S_m = S_{m+1} + |\mathcal{D} \cap k|$$

We finish this first part of the proof with the observation that the sum  $S_m$ , for any  $m \in \mathbb{Z}$ , can also be expressed as quotient

$$S_m = \frac{|\mathcal{D}^m \cap \mathcal{E}| - c_{\mathcal{E}}}{|k|_{\mathcal{E}}},$$

where  $c_{\mathcal{E}}$  is a constant independent of  $m$ : It is equal to the number of intersection points of  $\mathcal{D}$  with  $\mathcal{E}$  that are not contained in any of the arcs on some  $D_i \in \mathcal{D}$  which are parallel to an arc on  $k$ , where the two arcs intersect essentially, after possibly an isotopy of  $D_i$  or  $k$ .

Before starting with the second part of the proof, i.e. the comparison between the exceptional  $\mathcal{D}$ -values and the exceptional  $\mathcal{E}$ -values, let us observe the following properties of the exceptional values:

- (a) For every non-exceptional  $\mathcal{D}$ -value  $m \in \mathbb{Z}$  there is at least one of the curves  $D_i \in \mathcal{D}$  which intersects  $k$  essentially in some point  $P$  and satisfies  $tw_P(\delta_k^m(D_i), k) > 1$ . Thus  $D_i$  is carried by the

complete fat train track  $\tau$  with exceptional fibers  $\mathcal{E}_\tau = \mathcal{E}$  that is defined by  $k$  (compare Remark 3.6), and in particular  $D_i$  has no wave with respect to  $\mathcal{E}$ . Since  $\tau$  is maximal and the other  $D_j \in \mathcal{D}$  are disjoint from  $D_i$ , the previous statement is true for all of  $\mathcal{D}$ .

- (b) We stated above that every integer  $m \geq m_+ + 1$  satisfies  $S_m = S_{m-1} + |\mathcal{D} \cap k|$ . However, it is possible that the value  $m = m_+$  also satisfies this equality: This occurs if at every intersection point  $P$  of  $\delta_k^{m_+}(D_i)$  with  $k$ , for any  $D_i \in \mathcal{D}$ , one has:

$$tw_P(\delta_k^{m_+}(D_i), k) = 1$$

The analogous observation is true for  $m_-$  rather than  $m_+$ .

We conclude that there are two linear (or, rather, “affine”) functions  $m \mapsto S_+(m)$  and  $m \mapsto S_-(m)$  such that  $S_{m+\varepsilon} = S_+(m + \varepsilon)$  or  $S_{m+\varepsilon} = S_-(m + \varepsilon)$  for all  $\varepsilon \in \{-1, 0, 1\}$  and  $m \in \mathbb{Z} - N_{\mathcal{D}}$ , where  $N_{\mathcal{D}}$  is a subset of the  $\mathcal{D}$ -exceptional values  $m_-, \dots, m_+$ , and  $N_{\mathcal{D}}$  has either cardinality 3, 2, or 1. Furthermore, if  $N_{\mathcal{D}}$  has cardinality 1 and there are precisely three exceptional  $\mathcal{D}$ -values, then it is the middle one of those that is contained in  $N_{\mathcal{D}}$ .

(II.) By considering  $\mathcal{E}^m = \delta_k^{-m}(\mathcal{E})$  and the pairs  $\mathcal{D}, \mathcal{E}^m$  we observe that the hypothesis in the Proposition is symmetric in  $\mathcal{D}$  and  $\mathcal{E}$ , and that the same arguments as in part (I.) apply to  $\mathcal{E}$  rather than  $\mathcal{D}$ . We now note that  $|\mathcal{D}^m \cap \mathcal{E}| = |\mathcal{D} \cap \mathcal{E}^m|$ , since the homeomorphism  $\delta^{-m}$  preserves the intersection number. Furthermore, from the last equation in part (I.) before observation (a), we obtain the expression

$$|\mathcal{D}^m \cap \mathcal{E}| = |k|_{\mathcal{E}} S_m + c_{\mathcal{E}},$$

where  $|k|_{\mathcal{E}}$  and  $c_{\mathcal{E}}$  are constants independent of  $m$ . This allows us to conclude that the “non-linear” exceptional values must satisfy:

$$N_{\mathcal{D}} = N_{\mathcal{E}}$$

Thus we can deduce from observation (b) above that the union of the  $\mathcal{D}$ -exceptional and the  $\mathcal{E}$ -exceptional values consists of maximally 4 elements. Thus observation (a) above finishes the proof.  $\square$

**Remark 3.13.** From a careful analysis of the details of the proof of Proposition 3.12 it seems possible that, under the additional assumption that at some intersection point the  $\mathcal{D}$ -length of the intersection arc is not an integer, and similarly for the  $\mathcal{E}$ -length, in the statement of Proposition 3.12 one does not have to exclude four but only three or



even fewer exceptional Dehn twist exponents. Indeed, statement (a) of that proof may in fact be true also for some of the exceptional  $\mathcal{D}$ -values  $m_-, \dots, m_+$ .

#### 4. HEEGAARD SPLITTINGS

Let  $H$  be a 3-dimensional handlebody of genus  $g \geq 2$ , and let  $\Sigma = \partial H$  denote its boundary surface. The set  $\mathcal{D}(H)$  of isotopy classes of essential simple closed curves on  $\Sigma$  that bound a disk in  $H$  is a subset of  $\mathcal{C}^0(\Sigma)$ . It is the vertex set of what is called the *disk complex* of the handlebody  $H$ , contained as a subcomplex in  $\mathcal{C}(\Sigma)$ .

Similarly, we consider complete decomposing systems, up to isotopy in  $\Sigma$ , which bound disk systems in  $H$ , and denote the set of such isotopy classes by  $\mathcal{CDS}(H)$ . As in previous sections, we will sometimes omit the distinction between systems of curves and their isotopy classes, to make the notation easier.

The most prominent place where 3-dimensional handlebodies appear in topology are *Heegaard splittings* of 3-manifolds: Let  $M$  be a closed orientable 3-manifold, and let  $\Sigma \subset M$  be a *Heegaard surface* of genus  $g \geq 2$ . This means that  $M$  decomposes along  $\Sigma$  into two genus  $g$  handlebodies  $V$  and  $W$ , so that  $M = V \cup_{\Sigma} W$ .

The *distance* of a Heegaard splitting  $M = V \cup_{\Sigma} W$  is defined by

$$d(V, W) = \min\{d_{\mathcal{C}}(D, E) \mid D \in \mathcal{D}(V), E \in \mathcal{D}(W)\},$$

where  $d_{\mathcal{C}}$  denotes, as before, the distance in the curve complex  $\mathcal{C}(\Sigma)$  (see [6]).

**Remark 4.1.** Assume that  $M$  is irreducible, i.e., every embedded essential 2-sphere bounds a 3-ball in  $M$ . If  $M$  has a Heegaard splitting  $M = V \cup_{\Sigma} W$  with distance  $d_{\mathcal{C}}(V, W) = 0$ , then the splitting is called *stabilized*. A Heegaard splitting which satisfies  $d(V, W) \leq 1$  is called *weakly reducible*. Heegaard splittings with  $d(V, W) \geq 2$  have been termed by Casson-Gordon (see [3]) *strongly irreducible*. They play an important role in 3-manifold theory.

A well known and easy observation states the following:

**Remark 4.2.** Given a complete decomposing system

$$\mathcal{D} = \{D_1, \dots, D_{3g-3}\} \in \mathcal{CDS}(V)$$

for a handlebody  $V$ , then any other essential disk-bounding curve  $D \in \mathcal{D}(V)$  is either parallel to one of  $D_i$ , or  $D$  has a wave with respect to  $\mathcal{D}$  (i.e.,  $D$  contains a wave in one of the pair-of-pants  $P_j \subset \Sigma$  complementary to  $\mathcal{D}$ , compare Definition 3.2 (c)).

One should keep in mind, however, that a curve  $D$  on  $\Sigma = \partial V$  may well contain a wave with respect to some  $\mathcal{D} \in \mathcal{CDS}(V)$ , even if  $D$  does not bound a disk in  $V$ .

A complete decomposing system  $\mathcal{D} = \{D_1, \dots, D_{3g-3}\} \subset \Sigma$  is said to *have a wave with respect to a second complete decomposing system*  $\mathcal{E} \subset \Sigma$  if some of the  $D_i$  has a wave with respect to  $\mathcal{E}$ .

**Lemma 4.3** ([6], Lemma 1.3). *For every Heegaard splitting of a 3-manifold  $M = V \cup_\Sigma W$  there always exists a pair of complete decomposing systems  $\mathcal{D} \in \mathcal{CDS}(V)$  and  $\mathcal{E} \in \mathcal{CDS}(W)$  which have no waves with respect to each other.*

Consider a complete decomposing system  $\mathcal{D} \subset \Sigma$ , and an essential curve  $E \subset \Sigma$ . There always is an isotopy of  $E$  which makes  $E$  tight with respect to  $\mathcal{D}$ , i.e., it eliminates all inessential intersection points from  $E \cap \mathcal{D}$ , so that  $E$  is cut by  $\mathcal{D}$  into arcs which are either seams or waves (see Definition 3.2 and the preceding discussion). As in Section 3, we will consider these arcs only up to *isotopy*, by which we mean isotopy of the pair  $(\Sigma, \mathcal{D})$ .

Let  $\tau \subset \Sigma$  be a maximal train track, i.e., all connected components of  $\Sigma \setminus \tau$  are triangles. Let  $\mathcal{D}$  be a complete decomposing system in  $\Sigma$  which is carried by  $\tau$ . Then every connected component  $P$  (always a pair-of-pants !) of  $\Sigma \setminus \mathcal{D}$  contains precisely two of the triangles complementary to  $\tau$ . We consider the following two possibilities:

- (1) Every wave in  $P$  can be carried by  $\tau$ , while every seam can be isotoped into some of the  $I$ -fibers of  $\tau$ . We say that in this case  $P$  has  *$\Theta$ -graph* shape.
- (2) One of the seams and two non-isotopic waves can be carried by  $\tau$ , while a third wave as well as other two non-isotopic seams can be isotoped into some of the  $I$ -fibers of  $\tau$ . In this case  $P$  is said to have *eye glass* shape.

**Lemma 4.4.** *Let  $\tau \subset \Sigma$  be a maximal train track. Let  $\mathcal{D}$  be a complete decomposing system in  $\Sigma$  which is carried by  $\tau$ . Then every pair-of-pants  $P$  complementary to  $\mathcal{D}$  in  $\Sigma$  has either*

- (1)  *$\Theta$ -graph* shape, or
- (2) *eye glasses* shape.

*Proof.* Since  $\mathcal{D}$  is carried by  $\tau$ , the train track  $\tau$  defines an induced train track  $\tau_P = \tau \cap P$  on each pair-of-pants  $P$ , such that the  $I$ -bundle structure on  $\tau$  restricts to the  $I$ -bundle structure on  $\tau_P$ . It follows from an easy Euler characteristic count that each pair-of-pants complementary to  $\mathcal{D}$  must contain precisely two of the triangles that are complementary to  $\tau$ .

Consider now the deformation of the induced train track  $\tau_P$  onto the graph  $\Gamma_{\tau_P}$  as defined in Remark 2.1. There are two possible cases. The first, corresponding to case (1), is that  $\Gamma_{\tau_P}$  is a  $\Theta$ -graph. The second, corresponding to case (2), is that  $\Gamma_{\tau_P}$  is composed of two circles connected by an arc: it is an “eye glasses” graph.

Recall that a wave in  $P$  is an arc from one boundary component to itself which is not  $\partial$ -parallel, and that a seam in  $P$  is an arc connecting two different boundary components of  $P$ . It is easy to check that in case (1) all waves are carried by  $\tau_P$  and all seams are isotopic into the  $I$ -fibers, so that  $P$  has  $\Theta$ -graph shape. In case (2) one of the seams and two non-isotopic waves can be carried by  $\tau_P$ , while a third wave as well as other two non-isotopic seams can be isotoped into some of the  $I$ -fibers of  $\tau_P$ : the pair-of-pants  $P$  has an eye glasses shape. □

Notice that, contrary perhaps to the impression given above, there is more than one pair-of-pants, up to homeomorphisms that respect the complementary components and the singular  $I$ -fibration, which has eye glasses shape, and more than one which has  $\Theta$ -graph shape. Indeed, the homeomorphism type of such pair-of-pants depends also on the direction of the train track switches at the singular fibers.

**Remark 4.5.** Note that case (2) of Lemma 4.4 cannot occur for a complete fat train track  $\tau$  which has  $\mathcal{E}$  as exceptional fibers, if  $\mathcal{E}$  does not have waves with respect to  $\mathcal{D}$ : Indeed, waves which can be isotoped into the  $I$ -fibers of  $\tau$  are parallel to arcs on  $\mathcal{E}$  which then would also be waves with respect to  $\mathcal{D}$ .

Conversely, if each of the pair-of-pants complementary to  $\mathcal{D}$  has  $\Theta$ -graph shape, then  $\mathcal{E}$  does not have waves with respect to  $\mathcal{D}$ .

**Lemma 4.6.** *Let  $\tau \subset \Sigma$  be a complete fat train track, and let  $\tau' \subset \tau$  be a train track derived from  $\tau$ . Let  $\mathcal{D}$  be a complete decomposing system in  $\Sigma$  which is carried by  $\tau'$ , with the property that every pair-of-pants complementary to  $\mathcal{D}$  has  $\Theta$ -graph shape. Let  $D \subset \Sigma$  be an essential simple closed curve which is tight with respect to  $\mathcal{D}$ , and assume that some arc  $\beta$  from the set of arcs  $D - \mathcal{D}$  is a wave with respect to  $\mathcal{D}$ . Then  $D$  can be carried by  $\tau$ .*

*Proof.* Let  $P$  be the pair-of-pants complementary to  $\mathcal{D}$  that contains the wave  $\beta$ . By assumption  $P$  has  $\Theta$ -graph shape, so that  $\beta$  is carried by  $\tau'$ . We observe that any wave in such a  $\Theta$ -graph shaped pair-of-pants  $P$  has to run parallel on  $\tau'$  to at least one entire side of one of the two connected components complementary to  $\tau'$  which are contained in  $P$ . But this implies immediately that  $\beta$  contains a subarc  $\beta'$  that has to run parallel to some complete unzipping path which is used to derive  $\tau'$  from  $\tau$ , and thus  $\beta$  covers  $\tau$ . Note that  $\beta'$  has its endpoints on  $\mathcal{E}_\tau$ . The rest of the proof is now a direct application of Lemma 3.9, applied to the subarc of  $\beta'$  of  $\beta$ .  $\square$

**Theorem 4.7.** *Let  $M$  be an oriented 3-manifold with a Heegaard splitting  $M = V \cup_\Sigma W$ . Consider complete decomposing systems  $\mathcal{D} \in \mathcal{CDS}(V)$  and  $\mathcal{E} \in \mathcal{CDS}(W)$  which do not have waves with respect to each other. Let  $\tau \subset \Sigma$  be a complete fat train track with exceptional fibers  $\mathcal{E}_\tau = \mathcal{E}$ , and assume that  $\mathcal{D}$  is carried by  $\tau_n$ , for some  $n$ -tower of derived train tracks  $\tau = \tau_0 \supset \tau_1 \supset \dots \supset \tau_n$  with  $n \geq 2$ . Then the distance of the given Heegaard splitting satisfies:*

$$d(V, W) \geq n$$

*Proof.* By hypothesis the system  $\mathcal{D}$  is carried by  $\tau_n$ . Let  $D \in \mathcal{D}(V)$  be any disk-bounding essential simple closed curve in  $V$ . From Remark 4.2 we know that either  $D \in \mathcal{D}$ , or else  $D$  contains a wave with respect to  $\mathcal{D}$ . From Lemma 4.4 and Remark 4.5 we deduce that this wave is carried by  $\tau_n$ . Thus we can apply Lemma 4.6, and obtain that  $D$  is carried by  $\tau_{n-1}$ .

On the other hand, any essential disk-bounding simple closed curve  $E \in \mathcal{D}(W)$  in  $W$  has a wave with respect to  $\tau = \tau_0$ , by Definition 3.2 (d) and Remark 4.2. Thus Corollary 3.8 gives the desired inequality.  $\square$

## 5. APPLICATION TO 3-MANIFOLDS

In this section we derive some applications of Theorem 4.7. There are other results in the recent literature about 3-manifolds  $M$  with Heegaard splitting of large Hempel distance, for example [1], [4] and [13]. The advantage of our method, seems to us, is that it is very practical and allows in particular derivation of concrete lower bounds for the distance of Heegaard splittings in combinatorial terms.

### 5.A. Application 1: Heegaard diagrams.

We first describe a practical way how to derive a lower bound for the Hempel distance for a 3-manifold  $M$  given by a *Heegaard diagram*: The latter is given by a standardly embedded handlebody  $W \subset \mathbb{R}^3$  with boundary surface  $\Sigma = \partial W$ , equipped with a complete decomposing system  $\mathcal{D} \subset \Sigma$  that defines a second handlebody  $V$  with  $\Sigma = \partial V$ . The handlebody  $V$  (which usually cannot be embedded in  $\mathbb{R}^3$ ) is determined by the condition  $\mathcal{D} \in \mathcal{CDS}(V)$ , i.e.  $V$  contains a system of essential disks with boundary  $\mathcal{D}$ .

One first picks at random a complete decomposing system  $\mathcal{E} \in \mathcal{CDS}(W)$ . Then one modifies the pair  $(\mathcal{D}, \mathcal{E})$  iteratively to find complete decomposing systems  $(\mathcal{D}', \mathcal{E}')$  which bound disks in the same handlebodies  $V$  and  $W$  and, in addition, have no waves with respect to each other. Since the sets  $\mathcal{CDS}(V)$  and  $\mathcal{CDS}(W)$  are countable, Hempel's existence result for such systems given in Lemma 4.3 implies that there exists algorithms to find such systems  $(\mathcal{D}', \mathcal{E}')$ .

For practical purposes, an efficient procedure seems to be the passage from the complete decomposing systems  $(\mathcal{D}, \mathcal{E})$  (by omitting some of the curves from  $\mathcal{D}$  and  $\mathcal{E}$ ) to *minimal decomposing systems* that cut the surface  $\Sigma$  into a single simply connected complementary component, i.e., a  $2g$ -punctured 2-shere: One can then apply Whiteheads algorithm (see [7] and the references given there) to strictly reduce intersection number until the minimum is achieved. Adding the other disks back in, needed to make the minimal decomposing systems complete, gives good candidates for systems  $(\mathcal{D}', \mathcal{E}')$  that have no waves with respect to each other.

Once such systems  $(\mathcal{D}', \mathcal{E}')$  are found, one can easily determine a complete fat train track  $\tau$  with  $\mathcal{E}_\tau = \mathcal{E}'$  that carries  $\mathcal{D}'$ , by considering the intersection arcs on  $\mathcal{D}'$  with the pair-of-pants on  $\Sigma$  that are complementary to the system  $\mathcal{E}'$ , compare Lemma 3.5 and its proof. Next one starts splitting  $\tau_0 = \tau$  iteratively until every unzipping path is complete (or the weaker condition from Remark 2.10 is satisfied), to obtain the (weakly) derived train track  $\tau_1$  with respect to  $\mathcal{D}'$ . One then splits again  $\tau_1$ , to obtain a (weakly) derived train track  $\tau_2$ , and so on, until the first collision arises. The last derived train track  $\tau_n$  constructed before the collision gives the desired distance bound:

**Proposition 5.1.** *If a Heegaard splitting  $V \cup_{\Sigma} W$  is given by a Heegaard diagram as above, and if the above construction yields a train track  $\tau_n$  that carries  $\mathcal{D}'$ , then the Hempel distance of the Heegaard splitting satisfies:*

$$d(V, W) \geq n$$

*Proof.* This is an immediate consequence of the above construction and Theorem 4.7. □

### 5.B. Application 2: large distance via surgery.

We now describe a practical way how to construct 3-manifolds with Heegaard splittings of arbitrary high distance.

Let  $\Sigma$  be a surface of genus  $g \geq 2$ , and let  $\tau \subset \Sigma$  be a complete fat train track with associated complete decomposing system  $\mathcal{E} = \mathcal{E}_{\tau}$ . One successively derives (or weakly derives) train tracks  $\tau_0 = \tau, \tau_1, \tau_2, \dots, \tau_n$ , and then chooses a curve  $k \subset \Sigma$  that covers  $\tau_n$ . We note that, since  $k$  is carried by  $\tau_n$  and hence by  $\tau_0$ , it does not contain waves with respect to  $\mathcal{E}$ . Let  $\delta_k$  denote the Dehn twist on  $\Sigma$  along the curve  $k$ .

**Proposition 5.2.** *The Heegaard splitting  $V \cup_{\Sigma} W$ , defined by the two complete decomposing systems  $\mathcal{E}$  and  $\mathcal{D} = \delta_k^m(\mathcal{E})$  on  $\Sigma$  via the conditions  $\mathcal{D} \in \mathcal{CDS}(V)$  and  $\mathcal{E} \in \mathcal{CDS}(W)$ , has distance*

$$d(V, W) \geq n,$$

for any integer  $m \in \mathbb{Z} \setminus \{1, 0, -1\}$ .

*Proof.* We first observe that for any  $D \in \mathcal{D}$  and any intersection point  $P \in D \cap k$  the twisting number of  $D$  along  $k$  at  $P$  satisfies  $tw_P(D, k) = 0$ . Hence one can apply Lemma 3.10 (a) directly to deduce that  $tw_P(\delta_k^m(D), k) = |m|$ , for any  $m \in \mathbb{Z}$ . Thus Lemma 3.10 (b) yields

that for every value of  $m \in \mathbb{Z} \setminus \{1, 0, -1\}$  the curve system  $\mathcal{D} = \delta_k^m(\mathcal{E})$  is carried by  $\tau_n$ . In particular, it follows that  $\mathcal{D}$  does not contain waves with respect to  $\mathcal{E}$ , and by symmetry of the construction (i.e.  $\mathcal{E} = \delta_k^{-m}(\mathcal{D})$ ), the vice versa assertion is also true. Hence Theorem 4.7 applies to give the claimed inequality. □

The above proof of Proposition 5.2 should be compared to the proofs of both, Theorem 3.1 of [13] and Theorem 1.1 of [4].

It is a well known fact that changing the gluing map of a given Heegaard splitting  $M = V \cup_{\Sigma} W$  by an  $m$ -fold Dehn twist along a curve  $k \subset \Sigma$  is equivalent to performing  $\frac{1}{m}$ -Dehn surgery on  $k$  along the slope determined by  $\Sigma$  on a regular neighborhood  $N(k)$ . We call this a  $\Sigma$ -horizontal  $\frac{1}{m}$ -surgery on  $k$  (compare [10]), and the obtained manifold is denoted by  $M_k^{\Sigma}(\frac{1}{m})$ .

In the situation considered in Proposition 5.2 the manifold  $M$  in question is a connected sum of  $g$  copies of  $S^1 \times S^2$ , provided with the standard Heegaard splitting of genus  $g$ . We will extend the construction in the next subsection to obtain an analogous result for knots  $k$  in  $S^3$ . Note, however, that in the case where  $M$  is the connected sum of copies of  $S^1 \times S^2$ , the alternative method described in [13] does not apply without additional modifications.

### 5.C. Application 3: surgery on knots in $S^3$ .

Let  $S^3 = V \cup_{\Sigma} W$  be the standard Heegaard decomposition of the 3-sphere of genus  $g \geq 2$ . Let  $\mathcal{D} \in \mathcal{CDS}(V)$  and  $\mathcal{E} \in \mathcal{CDS}(W)$  be complete decomposing systems. Let  $c \subset \Sigma$  be a curve that has no waves with respect to either  $\mathcal{D}$  or  $\mathcal{E}$ , and which fills every pair-of-pants  $P$  complementary to either  $\mathcal{D}$  or  $\mathcal{E}$  (compare Remark 3.6). Examples for such  $\mathcal{D}, \mathcal{E}$  and  $c$  are not hard to find; see [10] or [13] for such examples with arbitrary large  $g = \text{genus}(\Sigma)$ . We denote by  $\tau_{\mathcal{D}}(c)$  and  $\tau_{\mathcal{E}}(c)$  the two maximal train tracks that carry  $c$  and have  $\mathcal{D}$  and  $\mathcal{E}$  respectively as exceptional fibers.

We now consider an arbitrary minimal-maximal lamination  $\mathcal{L} \subset \Sigma$ . We deduce from Lemma 3.10 and Remark 3.11 (b) that for all sufficiently large integers  $t \geq 0$  the Dehn twist  $\delta_c^t$  gives a lamination  $\mathcal{L}' = \delta_c^t(\mathcal{L})$  that covers both,  $\tau_{\mathcal{D}}(c)$  and  $\tau_{\mathcal{E}}(c)$ . In particular,  $\mathcal{L}'$  has no wave with respect to either  $\mathcal{D}$  or  $\mathcal{E}$ .

Next one derives successively train tracks  $\tau_0 = \tau_{\mathcal{E}}(c), \tau_1, \tau_2, \dots, \tau_n$  with respect to  $\mathcal{L}'$  and then chooses a curve  $k$  on  $\Sigma$  that covers both,  $\tau_n$  and  $\tau_{\mathcal{D}}(c)$ . Any curve that is sufficiently close to  $\mathcal{L}'$  (in the space of projective measured laminations) will have this property. We note that, since  $k$  covers  $\tau_n$  and hence also  $\tau_0$ , it fills every pair-of-pants complementary to  $\mathcal{E}$  and does not contain waves with respect to  $\mathcal{E}$ . Similarly, since  $k$  covers  $\tau_{\mathcal{D}}(c)$ , it fills every pair-of-pants complementary to  $\mathcal{D}$  and does also not contain waves with respect to  $\mathcal{D}$ . Thus

all conditions of Proposition 3.12 are satisfied, so that the resulting systems  $\mathcal{D}^m = \delta_k^m(\mathcal{D})$  and  $\mathcal{E}$  do not have waves with respect to each other, and  $\mathcal{D}^m$  is carried by  $\tau_n$ . Thus we can apply Theorem 4.7.

It remains to recall the above observation that the  $m$ -fold Dehn twist at  $k$  amounts precisely to  $\Sigma$ -horizontal  $\frac{1}{m}$ -surgery on  $k$ .

Thus we have proved the following result, which has been stated in a slightly weakened form at the beginning of the Introduction:

**Theorem 5.3.** *For any integer  $n \geq 0$  there exist a knot  $k \in S^3$  and an integer  $m_0$ , such that for any  $m \in \mathbb{Z} \setminus \{m_0, m_0 + 1, m_0 + 2, m_0 + 3\}$  the 3-manifold  $M_k^\Sigma(\frac{1}{m})$ , obtained by  $\Sigma$ -horizontal  $\frac{1}{m}$ -surgery on  $k$ , admits a Heegaard splitting  $M_k^\Sigma(\frac{1}{m}) = V \cup_\Sigma W$  which satisfies*

$$d(V, W) \geq n.$$

□

We would like to comment on the relationship between this result and Theorem 3.1 of [13]: If a Heegaard splitting  $M = V \cup_\Sigma W$  is obtained through Dehn filling along a curve  $k \subset \Sigma \subset \overset{\circ}{M}$  from a Heegaard splitting of the knot exterior  $E(k) = M \setminus \overset{\circ}{N}(k)$ , then the distance of the latter splitting is bounded below by  $d(V, W)$ . The converse inequality, however, does not hold. On the other hand, if  $k$  is not primitive on either  $V$  or  $W$  (as may well occur in our examples above), then there is no Heegaard splitting of  $E(k)$  that induces the given splitting of  $M$ . Thus, although mathematically close, there is no direct implication either from Theorem 3.1 of [13] to the above Theorem 5.3, nor conversely.

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